may not know about it. The subscription list was doubled in 1913. If it should be doubled again in 1914, the journal would become self-sustaining. **It can be done.**

THE TACTICAL PROBLEM OF STEINER.

By W. H. BUSSEY, University of Minnesota.

NOTE BY THE EDITORS.—This article illustrates the reference in the editorial of this issue concerning "papers of a somewhat more technical character in which, however, we have tried to have the technical terms explained for the benefit of the general reader." Professor Bussey has met this request most admirably.

The study of tactical configurations known as triple systems had its origin in two related problems proposed independently by T. P. Kirkman[1] and J. Steiner.[2] Kirkman's problem is to arrange fifteen school girls in parties of three for seven consecutive days' walk so that every two of the girls walk together once and only once during the seven days. There is a good account of the history of the problem with several methods of solution in Ball's *Mathematical Recreations and Essays*, 5th edition, Chapter 9.

An arrangement of a number of elements in sets of three so that every set of two is contained in one and only one set of three is called a triple system. The sets of three are called triples or triads. The problem of the fifteen school girls involves a triple system of 15 elements and 35 triads. The simplest triple system is the following well-known one of 7 elements and 7 triads. The digits 0, 1, 2, 3, 4, 5, 6 are the elements and the columns are the triads.

0	1	2	3	4	5	6
1	2	3	4	5	6	0
3	4	5	6	0	1	2

The seven elements of this triple system can be arranged in sets of four, called tetrads, so that no triad is contained in a tetrad and so that every set of three which is not a triad is contained in one and only one tetrad. The arrangement is as follows.

0	1	2	3	4	5	6
1	2	3	4	5	6	0
2	3	4	5	6	0	1
5	6	0	1	2	3	4

It is well known that the nine points of inflexion of a plane cubic curve lie by threes on twelve straight lines. Four lines pass through each point of inflexion. Any two of the nine points thus determine a third, and the nine points form a

[1] *The Lady's and Gentleman's Diary*, 1850.
[2] *Journal für die reine und angewandte Mathematik*, Vol. 45, pp. 181–182.

triple system with twelve triads. The triple system of nine elements may be written as follows.

0	0	0	0	1	1	1	2	2	2	3	6
1	3	4	5	3	4	5	3	4	5	4	7
2	6	7	8	7	8	6	8	6	7	5	8

These triple systems of seven and nine elements are important in the theory of equations of the 7th and 9th degrees.[1]

The problem proposed by Steiner is as follows. It was suggested to him by an investigation of the configuration of the 28 double tangents of a plane quartic curve.

For what values of n is it possible to arrange n elements in sets of three, called triples or triads, so that every set of two elements is contained in one and only one triad? If n is a number for which there is such an arrangement in triads, are there other arrangements which cannot be obtained from it by a mere permutation of the elements? When such an arrangement has been made, is it possible to arrange the n elements in sets of four, called tetrads, so that no triad is contained in a tetrad and so that every set of three which is not a triad is contained in one and only one tetrad? When such an arrangement in tetrads has been made, is it possible to arrange the n elements in sets of five, called pentads, so that no triad or tetrad is contained in a pentad, and so that every set of four which is not a tetrad and does not contain a triad is contained in one and only one pentad? When these successive arrangements have been made, up to and including an arrangement in *k-ads*, is it possible to arrange the n elements in sets of $k + 1$, called $(k + 1)$-*ads*, so that no *l-ad*, $l \leqq k$, is contained in a $(k + 1)$-*ad*, and so that every set of k elements which is not a *k-ad* and does not contain an *l-ad*, $l < k$, is contained in one and only one $(k + 1)$-*ad*?

The part of the problem that relates to triads has been completely solved.[2] This means that all of Steiner's questions have been answered. It does not mean that the last word on triple systems has been said or that all problems connected with triple systems have been solved.

Every pair of elements of a triple system determines a triad. The number of pairs that can be chosen from n elements is $\frac{1}{2}n(n-1)$. But each triad is determined by any one of three pairs. Therefore, to obtain the number of triads, divide the number of pairs by three. The result is $\frac{1}{6}n(n-1)$. The total number of elements in the triple system, counting duplicates, is three times the number of triads, namely $\frac{1}{2}n(n-1)$. Therefore each of the n elements occurs $\frac{1}{2}(n-1)$ times. This number must be an integer, and therefore n must be an odd number. It must therefore be of one of the forms $6m + 1, 6m + 3, 6m + 5$. But $\frac{1}{6}n(n-1)$ must also be an integer. This can happen when n is $6m + 1$ or $6m + 3$, but not when n is $6m + 5$. Therefore a necessary condition that n elements can be ar-

[1] See Netto's *Theory of Substitutions* (English translation by F. N. Cole), pp. 229–239, and Netto's *Vorlesungen über Algebra*, Vol. 2, pp. 460–480.

[2] See *Encyclopédie des Sciences Mathématiques*, Vol. 1, p. 80.

ranged to form a triple system is that n be of the form $6m + 1$ or $6m + 3$. It has been proved that this condition is also sufficient. When $n = 7$ or 9, there is essentially only one arrangement of the n elements in triads. Any two arrangements differ only in notation. When $n = 13$, there are two and only two essentially different triple systems. When n is any number of the form $6m + 1$ or $6m + 3$ and is greater than 13, there are at least two essentially different triple systems.[1] The two triple systems of 13 elements are compared in two recent papers by F. N. Cole[2] and H. S. White.[3]

This paper has to do primarily with the part of the Steiner problem that relates to tetrads, pentads, etc. If an arrangement of n elements in triads and tetrads is possible, the number of tetrads can be counted as follows. Every tetrad is determined by a set of three which is not a triad. The total number of sets of three that can be chosen from n elements is $\frac{1}{6}n(n-1)(n-2)$. The number of these which are triads has been found to be $\frac{1}{6}n(n-1)$. The difference between these two numbers, namely $\frac{1}{6}n(n-1)(n-3)$, is the number of sets of three which determine tetrads. But each tetrad can be determined by three elements in as many ways as there are combinations of four things three at a time, namely in four ways. Therefore, to get the number of tetrads, divide by four. The result is $\frac{1}{24}n(n-1)(n-3)$. When $n = 7$, this number is also equal to 7. An arrangement of seven elements in seven tetrads has already been given.

If there is an arrangement of n elements in triads, tetrads, pentads, etc., the number of k-*ads* for $k = 3, 4, 5$, etc., is given by the formula

$$N_k = \frac{1}{k!}n(n-1)(n-3)(n-7)\cdots(n-[2^{k-2}-1]),$$

which was given by Steiner without proof. It can be proved as follows. Let $a_1, a_2, a_3, \cdots, a_{k-1}$ be a set of $k-1$ elements that is not a $(k-1)$-*ad* and does not contain an l-*ad*, $l < k-1$. Such a set determines a k-*ad*. The number of such sets is k times the number of k-*ads*, namely kN_k, because the same k-*ad* is determined by any one of k different sets of $k-1$ elements each. Every $(k+1)$-*ad* is determined by a set of k elements that is not a k-*ad* and does not contain an l-*ad*, $l < k$. Any $k-1$ elements of such a set constitute a set $a_1, a_2, a_3, \cdots, a_{k-1}$ of the kind mentioned above. Therefore, to get a set of k elements to determine a $(k+1)$-*ad*, adjoin to these an element a_k which is such that it will not combine with any of the others to form an l-*ad* for any $l \leqq k$. In choosing a_k, the following elements must be avoided.

1. The ${}_{k-1}C_1$ elements $a_1, a_2, a_3, \cdots, a_{k-1}$.
2. The ${}_{k-1}C_2$ elements that form triads with the elements $a_1, a_2, a_3, \cdots, a_{k-1}$ taken in pairs.

[1] For proofs of these statements see Moore, *Concerning Triple Systems*, *Mathematische Annalen*, Vol. 43, pp. 271–285, and Netto, *Vorlesungen über Algebra*, Vol. 2, p. 474.

[2] *Transactions of the American Mathematical Society*, Vol. 14 (1913), pp. 1–5.

[3] *Transactions of the American Mathematical Society*, Vol. 14 (1913), pp. 6–13.

3. The ${}_{k-1}C_3$ elements that form tetrads with the elements $a_1, a_2, a_3, \cdots, a_{k-1}$ taken in threes.

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$k-1$. The ${}_{k-1}C_{k-1}$ elements that form k-*ads* with the elements $a_1, a_2, a_3, \cdots, a_{k-1}$ taken $k-1$ at a time.

The total number of elements to be avoided is therefore

$$ {}_{k-1}C_1 + {}_{k-1}C_2 + {}_{k-1}C_3 + \cdots + {}_{k-1}C_{k-1}, $$

which is the number of combinations of $k-1$ things taken any number at a time, and is equal to $2^{k-1} - 1$.[1] The number of ways in which the a_k can be chosen is therefore $n - (2^{k-1} - 1)$, and the number of ways in which k elements can be chosen to determine a $(k+1)$-*ad* is apparently $kN_k[n - (2^{k-1} - 1)]$. But the set of k elements has been obtained by adjoining an element a_k to a certain sub-set $a_1, a_2, a_3, \cdots, a_{k-1}$. It can be obtained in the same way by adjoining an element to any other sub-set of $k-1$ of the same k elements. These sub-sets are ${}_kC_{k-1} = k$ in number. Therefore the count is k times as large as it ought to be; and the number of ways in which a set of k elements can be chosen to determine a $(k+1)$-*ad* is $N_k[n - (2^{k-1} - 1)]$. But the same $(k+1)$-*ad* is determined by any k of its elements, that is in ${}_{k+1}C_k = k+1$ ways. Therefore the total number of $(k+1)$-*ads* is $N_k\left[\dfrac{n - (2^{k-1} - 1)}{k+1}\right]$. In other words, to obtain the number of $(k+1)$-*ads*, multiply the number of k-*ads* by $\dfrac{n - (2^{k-1} - 1)}{k+1}$. The number of tetrads has already been found to be $\dfrac{1}{4!}n(n-1)(n-3)$. Therefore the number of pentads is $\dfrac{1}{5!}n(n-1)(n-3)(n-7)$; and the number of k-*ads* for any value of k is that given by Steiner's formula.

In the *Bulletin of the American Mathematical Society*, Vol. 16, pp. 19–22, the author of this paper proved that when n is a number of the form $2^j - 1$ there exists an arrangement of n elements in triads, tetrads, pentads, etc., up to and including $(j+1)$-*ads*. That paper made use of finite projective geometries of many dimensions.[2] The present paper gives the solution in a more simple way which makes no use of geometry. It not only proves that the arrangement is possible, but also gives a method by which the arrangement can easily be written down in any particular case. The paper makes use of the theory of linear dependence in a finite field.

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[1] See Fine's *College Algebra*, §770.

[2] See Veblen and Bussey, "Finite Projective Geometries," *Transactions of the American Mathematical Society*, Vol. 7, pp. 241–259.

Note.—M. G. Brunel, in *Procès-Verbaux, Soc. des Sciences de Bordeaux*, 1896–1897, pp. 37–41, gave an arrangement of 9 elements in triads and tetrads; one of 7 elements in triads and tetrads; and one of 15 elements in triads, tetrads, and pentads. He stated that the method which he used in the last two cases could be extended to the case of $2^j - 1$ elements. His paper gave merely the results in the three cases ($n = 7, 9, 15$) without any suggestion of method.

The Finite Algebra of Order p.

Any system of s distinct symbols or marks which can be combined by four operations that satisfy the same formal laws as the four fundamental operations of algebra, addition, subtraction, multiplication, and division, so that when the marks are so combined the result of the operations is in every case unique and belongs to the given system of marks, are said to constitute a "finite algebra" or "field" of order s.

In the theory of numbers the integers $0, 1, 2, \cdots, (p-1)$ are said to constitute a system of "least residues," modulo p. No two of them are congruent to each other, and every other integer is congruent to some one of them, modulo p. The sum of two of them, say a and b, is congruent to some one of them, say c. Then c is said to be the "sum of a and b modulo p." The operation of finding c when a and b are given is called "addition modulo p." Similarly "subtraction modulo p" and "multiplication modulo p" are defined. It is proved in the theory of numbers that, when p is a prime, there is one and only one least residue c of p, such that $ac \equiv b$, modulo p, where b is any least residue of p and a is any one except zero. c is called the quotient of b divided by a modulo p, and the operation of finding c when b and a are given is called "division modulo p." Methods of finding c in any given case are given in books on the theory of numbers. It is also proved in the theory of numbers that these operations which are called addition, subtraction, multiplication, and division, modulo p, satisfy the same formal laws as the corresponding operations of ordinary algebra. Therefore the least residues modulo p, where p is a prime, constitute a "finite algebra" or "field" of order p.

Linear Dependence in the Field of Order p.

In the field of order p, m sets of n marks each

$$(A) \qquad \begin{array}{l} x_1', \quad x_2', \quad x_3', \quad \cdots, x_n' \\ x_1'', \quad x_2'', \quad x_3'', \quad \cdots, x_n'' \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_1^{[m]}, \quad x_2^{[m]}, \quad x_3^{[m]}, \quad \cdots, x_n^{[m]} \end{array}$$

are said to be linearly dependent when there exist in the field m marks $c', c'', c''', \cdots, c^{[m]}$, not all zero, such that

$$c'x_j' + c''x_j'' + c'''x_j''' + \cdots + c^{[m]}x_j^{[m]} \equiv 0, \text{ mod. } p \quad (j = 1, 2, 3, \cdots, n).$$

If this is not the case, the sets are said to be linearly independent.

This definition is analogous to the definition of linear dependence in ordinary algebra.[1] The proofs of the following theorems are almost word for word like the proofs of the corresponding theorems in ordinary algebra, and for that reason are not given here.

THEOREM 1. Two sets of n marks each, not all zero, are linearly dependent when and only when they are proportional.

THEOREM 2. If there exist among m sets of marks a smaller number of sets which are linearly dependent, then the m sets are linearly dependent.

THEOREM 3. If any one of the m sets of marks consists wholly of zeros, the m sets are linearly dependent.

THEOREM 4. A necessary and sufficient condition that the m sets of marks (A) be linearly dependent when $m \leqq n$ is that all the m-rowed determinants of the matrix

$$\left\| \begin{matrix} x_1' & x_2' & x_3' & \cdots & x_n' \\ x_1'' & x_2'' & x_3'' & \cdots & x_n'' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1^{[m]} & x_2^{[m]} & x_3^{[m]} & \cdots & x_n^{[m]} \end{matrix} \right\|$$

be congruent to zero.

THEOREM 5. m sets of n marks each are always linearly dependent when $m > n$.

LINEAR DEPENDENCE IN THE FIELD OF ORDER 2.

In the field of order 2, the only marks are 0 and 1, and therefore, by Theorem 1, two sets of n marks each, not all zero, are linearly dependent when and only when they are identical.

Consider m different sets of n marks each, not all zero, in the field of order 2, $m \leqq n$,

$$(B) \qquad \begin{matrix} x_1', & x_2', & x_3', & \cdots, & x_n' \\ x_1'', & x_2'', & x_3'', & \cdots, & x_n'' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1^{[m]}, & x_2^{[m]}, & x_3^{[m]}, & \cdots, & x_n^{[m]} \end{matrix}$$

such that the m sets are linearly dependent and such that no q of the sets are linearly dependent for any $q < m$. Then there are in the field m marks c', c'', c''', $\cdots$, $c^{[m]}$, not all zero, such that

$$c'x_j' + c''x_j'' + c'''x_j''' + \cdots + c^{[m]}x_j^{[m]} \equiv 0, \bmod 2 \quad (j = 1, 2, 3, \cdots, n).$$

[1] For an account of the theory of linear dependence in ordinary algebra, see Bôcher's "Introduction to Higher Algebra," Chapter III.

No one of the c's can be zero; for if one of them were, say $c^{[m]}$, this relation would become

$$c'x_j' + c''x_j'' + c'''x_j''' + \cdots + c^{[m-1]}x_j^{[m-1]} \equiv 0, \text{ mod } 2 \quad (j = 1, 2, 3, \cdots, n),$$

where the c's are not all zero, which would prove that the first $m - 1$ sets are linearly dependent. But this is contrary to the hypothesis about the sets (B). Therefore every one of the c's must be 1, and the following theorem is proved.

THEOREM 6. If m sets of n marks each, not all zero, in the field of order 2, $m \leqq n$, are linearly dependent, and if no q of them, $q < m$, are linearly dependent, then

$$x_j' + x_j'' + x_j''' + \cdots + x_j^{[m]} \equiv 0, \text{ mod } 2 \quad (j = 1, 2, 3, \cdots, n).$$

THEOREM 7. Any one of the marks $x_j^{[i]}$, $[i = 1, 2, \cdots, m]$, of Theorem (6), is congruent to the sum of all the others, modulo 2.

This is because, in the algebra of integers modulo 2, the mark -1 is congruent to the mark $+1$, and consequently any term of a congruence can be transposed from one side of the congruence to the other.

Let the following be $m - 1$ linearly independent sets of marks of the field of order 2.

$$\begin{array}{cccccc}
x_1', & x_2', & x_3', & \cdots, & x_n' \\
x_1'', & x_2'', & x_3'', & \cdots, & x_n'' \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
x_1^{[m-1]}, & x_2^{[m-1]}, & x_3^{[m-1]}, & \cdots, & x_n^{[m-1]}
\end{array}$$

Adjoin to these a new set $x_1^{[m]}, x_2^{[m]}, \cdots, x_n^{[m]}$, where $x_j^{[m]}$, $(j = 1, 2, 3, \cdots, n)$, is the mark of the field of order 2 determined by the congruence

$$x_j^{[m]} \equiv x_j' + x_j'' + x_j''' + \cdots + x_j^{[m-1]}, \text{ mod } 2, \tag{1}$$

which can also be written in the form

$$x_j' + x_j'' + x_j''' + \cdots + x_j^{[m-1]} + x_j^{[m]} \equiv 0, \text{ mod } 2.$$

This last relation proves that the m sets are linearly dependent. It can be proved that no $m - 1$ of the m sets are linearly dependent. The proof is as follows. The first $m - 1$ sets are linearly independent by hypothesis, and therefore, by Theorem 4, at least one of the $(m - 1)$-rowed determinants of the matrix

$$\left\| \begin{array}{ccccc}
x_1' & x_2' & x_3' & \cdots & x_n' \\
x_1'' & x_2'' & x_3'' & \cdots & x_n'' \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
x_1^{[m-1]} & x_2^{[m-1]} & x_3^{[m-1]} & \cdots & x_n^{[m-1]}
\end{array} \right\|$$

is not congruent to zero. It can be assumed without loss of generality that the sets have been so arranged that the determinant of the first $m-1$ columns is one not congruent to zero; that is, the determinant

$$(C) \qquad \begin{vmatrix} x_1' & x_2' & x_3' & \cdots & x_{m-1}' \\ x_1'' & x_2'' & x_3'' & \cdots & x_{m-1}'' \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ x_1^{[m-1]} & x_2^{[m-1]} & x_3^{[m-1]} & \cdots & x_{m-1}^{[m-1]} \end{vmatrix}$$

is not congruent to zero. Now consider $m-1$ of the m sets including the m-th set, and suppose the one omitted to be the first set. The necessary and sufficient condition that they be linearly dependent is that all the $(m-1)$-rowed determinants of the matrix

$$\begin{Vmatrix} x_1'' & x_2'' & x_3'' & \cdots & x_n'' \\ x_1''' & x_2''' & x_3''' & \cdots & x_n''' \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ x_1^{[m-1]} & x_2^{[m-1]} & x_3^{[m-1]} & \cdots & x_n^{[m-1]} \\ x_1^{[m]} & x_2^{[m]} & x_3^{[m]} & \cdots & x_n^{[m]} \end{Vmatrix}$$

be congruent to zero. The $(m-1)$-rowed determinant of the first $m-1$ columns is

$$(D) \qquad \begin{vmatrix} x_1'' & x_2'' & x_3'' & \cdots & x_{m-1}'' \\ x_1''' & x_2''' & x_3''' & \cdots & x_{m-1}''' \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ x_1^{[m]} & x_2^{[m]} & x_3^{[m]} & \cdots & x_{m-1}^{[m]} \end{vmatrix}$$

Replace each element of the last row by its value as given by relation (1), and the determinant becomes

$$\begin{vmatrix} x_1'' & x_2'' & \cdots & x_{m-1}'' \\ x_1''' & x_2''' & \cdots & x_{m-1}''' \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ x_1^{[m-1]} & x_2^{[m-1]} & \cdots & x_{m-1}^{[m-1]} \\ (x_1'+x_1''+\cdots+x_1^{[m-1]}), & (x_2'+x_2''+\cdots+x_2^{[m-1]}), & \cdots & (x_{m-1}'+x_{m-1}''+\cdots+x_{m-1}^{[m-1]}) \end{vmatrix}$$

By the well-known theorem about the addition of determinants, this determinant is equal to the sum of $m - 1$ determinants, the first of which is

$$\begin{vmatrix} x_1'' & x_2'' & \cdots & x_{m-1}'' \\ x_1''' & x_2''' & \cdots & x_{m-1}''' \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{[m-1]} & x_2^{[m-1]} & \cdots & x_{m-1}^{[m-1]} \\ x_1' & x_2' & \cdots & x_{m-1}' \end{vmatrix}$$

The others are all congruent to zero, because in each one of them two rows are identical. The one determinant is not congruent to zero because it differs from the determinant (C) only in the order of its rows. Therefore the determinant (D) is not congruent to zero; and the $m - 1$ sets under consideration are not linearly dependent. It has now been proved that the $m - 1$ sets obtained by omitting the first of the m sets are linearly independent. The argument holds good when the omitted set is not the first one but any other one. This completes the proof that no $m - 1$ of the m sets are linearly dependent. By Theorem 2, it follows that no q of the m sets, $q < m$, are linearly dependent. The result may be stated as the following theorem.

THEOREM 8. Any $m - 1$ sets of n marks each, not all zero, in the field of order 2, which are linearly independent determine an m-th set so that the m sets are linearly dependent and so that no q of them, $q < m$, are linearly dependent.

APPLICATION TO STEINER'S PROBLEM.

Consider the $2^j - 1$ different elements of the form $(x_1, x_2, x_3, \cdots, x_j)$, each x being a mark of the field of order 2, that is, either 0 or 1, the element $(0, 0, 0, \cdots, 0)$ being excluded. Any two of them are linearly independent because they are different. Any three which are linearly dependent are said to constitute a triad. By Theorem 8, any two elements determine a third to form a triad. If the two elements are $(x_1, x_2, x_3, \cdots, x_j)$ and $(y_1, y_2, y_3, \cdots, y_j)$, the third is $(x_1 + y_1, x_2 + y_2, x_3 + y_3, \cdots, x_j + y_j)$, where the $+$ signs mean addition modulo 2. Also by Theorem 8, any three elements which are linearly independent, and therefore do not form a triad, determine a fourth element so that the four are linearly dependent but no three of them are linearly dependent. Such a set of four is called a tetrad. Any set of four elements which are linearly independent is not a tetrad and, by Theorem 2, does not contain a triad. Such a set of four determines a fifth element, by Theorem 8, such that the five are linearly dependent but also such that no four or three are linearly dependent. Such a set of five is called a pentad. Any set of k elements which are linearly independent is not a *k-ad* and, by Theorem 2, does not contain an *l-ad*, $l < k$. The k elements determine a $(k + 1)$-st element, by Theorem 8, such that the $k + 1$ elements are linearly dependent but also such that no l of them, $l < k + 1$, are linearly de-

pendent. Such a set of $k+1$ elements is called a $(k+1)$-*ad*. This arrangement in triads, tetrads, etc., is just the arrangement called for in the Steiner problem. There is no arrangement of 2^j+1 elements in *k-ads* for $k > j+1$ because, by Theorem 5, there is no set of $j+1$ linearly independent elements. Furthermore, the Steiner formula gives $N_k = 0$ when $k > j+1$. But there is an arrangement in *k-ads* for every $k \leqq j+1$.

As an example of the foregoing theory, the arrangement of seven elements in triads and tetrads is worked out as follows. In this case $j = 3$, and the $2^j - 1 = 7$ elements are

(001) (010) (101) (011) (111) (110) (100).

For convenience let these be denoted by the letters a, b, c, d, e, f, g, respectively. Any two elements (x_1, x_2, x_3) and (y_1, y_2, y_3) determine the third element $(x_1+y_1, x_2+y_2, x_3+y_3)$, where the + signs mean addition modulo 2. The three form a triad. Thus the elements a and b determine the element (001), which is the element d. Therefore one triad of the system is *abd*. Similarly all the other triads can be worked out. The complete arrangement is as follows, the columns being triads.

a	*b*	*c*	*d*	*e*	*f*	*g*
b	*c*	*d*	*e*	*f*	*g*	*a*
d	*e*	*f*	*g*	*a*	*b*	*c*

Let (x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3) denote any three elements that do not form a triad. They determine as the fourth element to form a tetrad the element $(x_1+y_1+z_1, x_2+y_2+z_2, x_3+y_3+z_3)$, where the + signs mean addition modulo 2. Thus the three elements a (001); b (010); and c (101) determine the element (110), which is the element f. Therefore one tetrad of the system is *abcf*. The others can be found in the same way. The complete arrangement is as follows, the columns being the tetrads.

a	*b*	*c*	*d*	*e*	*f*	*g*
b	*c*	*d*	*e*	*f*	*g*	*a*
c	*d*	*e*	*f*	*g*	*a*	*b*
f	*g*	*a*	*b*	*c*	*d*	*e*

This arrangement in triads and tetrads is, apart from notation, the same as the one given earlier in the paper.

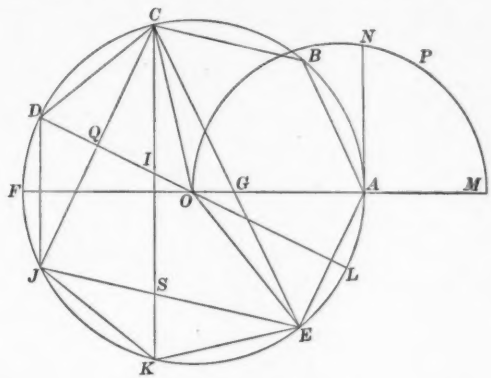
The following table gives the number of triads, tetrads, etc., when $n = 7, 15, 31, 63$.

n.	N_3.	N_4.	N_5.	N_6.	N_7.	N_8.
7	7	7	0	0	0	0
15	35	105	168	0	0	0
31	155	1,085	5,208	13,888	0	0
63	651	9,765	109,368	974,944	3,999,744	0

A GEOMETRICAL DISCUSSION OF THE REGULAR INSCRIBED HEPTAGON.

By J. Q. McNatt, Florence, Colo.

To calculate the length of the side of a regular inscribed heptagon when the radius is unity, suppose, in the accompanying figure, that the circumference is divided into seven equal parts at the points A, B, C, D, J, K, and E.



Draw the diameter DL and the chords JC, JE, CK, and CE and let CK cut JE at S.

Let h be the length of the side of the regular inscribed heptagon. It is proposed to compute h in terms of the radius as a unit. We have $DL : DC :: DC : DQ$, or in terms of the radius and h, $2 : h :: h : DQ$. That is, $DQ = h^2/2$. Moreover, $QL \times DQ = QC \times JQ$, and $QC = JQ$, or in terms of h we get

$$JQ = \sqrt{\left(2 - \frac{h^2}{2}\right)\left(\frac{h^2}{2}\right)} = \tfrac{1}{2}h\sqrt{4 - h^2}.$$

(1) Now $JE = JC = h\sqrt{4 - h^2}$, and $KE = SE$ since the two triangles KCE and KES are equiangular and hence similar, and since KCE is isosceles. Then in terms of h we find (2) $JS = JE - SE$, or $JS = h\sqrt{4 - h^2} - h$.

Now let DL cut KC at I and draw the radius CO. Then the two triangles OCD and CDI are equiangular and similar, and OCD is isosceles.

Then $CO : DC :: DC : DI$ or in terms of h, $1 : h :: h : DI$, from which $DI = h^2$. Also $DL - DI = IL = 2 - h^2$. Again, triangle DCI being isosceles, $DC = CI$, or $CI = h$. By geometry $IL \times DI = CI \times IK$, or in terms of h we have,

$$(2 - h^2)(h^2) = (h)(IK), \quad \text{or} \quad IK = 2h - h^3,$$

but $CI = h$, so that $KC = 3h - h^3$.

As the triangles KES and KCE are similar isosceles triangles, we have

$$CK : KE :: KE : SK,$$

or in terms of h,

$$3h - h^3 : h :: h : SK, \tag{3}$$

from which

$$SK = \frac{h}{3 - h^2}.$$

In the isosceles triangle JCS, $CS = JC = h\sqrt{4 - h^2}$ as shown above in (1). By geometry $CS \times SK = JS \times SE$, or in terms of h, using (1), (2) and (3) above,

$$(h\sqrt{4 - h^2})\left(\frac{h}{3 - h^2}\right) = (h\sqrt{4 - h^2} - h)(h). \tag{4}$$

Simplifying equation (4) we find

$$8 - 6h^2 + h^4 = (3 - h^2)\sqrt{4 - h^2}.$$

Factoring, dividing both members by $\sqrt{4 - h^2}$, squaring, and transposing, we get

$$7 - 14h^2 + 7h^4 - h^6 = 0, \tag{5}$$

which we call *The Heptagon Cubic.* Solving (5) by Horner's method we find,

$$h^2 = .7530203962821 \cdots,$$

or

$$h = .86776748 \cdots,$$

which is the side of the regular heptagon inscribed in a circle of radius unity.

Approximate Construction of a Heptagon.

From the similar isosceles triangles OEA and EAG, we get

$$OE : EA :: EA : GA, \tag{6}$$

or

$$1 : h :: h : GA,$$

from which

$$GA = h^2,$$

but

$$h^2 = .75 +, \text{ or } \tfrac{3}{4}OA.$$

Produce the line FA indefinitely and lay off on it the point G, so that

$$GA = \tfrac{3}{4}OA.$$

Then mark the point M, so that $AM = GA$.

With OM as a diameter construct the semicircle OPM and at the point A erect a perpendicular AN, which will be a mean proportional between OA and AM, that is, between OE and GA, so that by (6) $AN = EA$.

Then with a chord equal to the line AN lay off the points of the seven sided polygon, A, B, C, D, etc., which will be a regular inscribed heptagon, to a very close approximation.

A THEOREM IN NUMBER THEORY CONNECTED WITH THE BINOMIAL FORMULA.

By D. N. LEHMER, University of California.

The theorem is: *In the expansion of* $(P+Q)^n$, *where* P *and* Q *as well as* n *are integers, there will be a pair of consecutive equal terms when, and only when,* $n \equiv -1 \bmod (P+Q)/\delta$, *where* δ *is the greatest common divisor of* P *and* Q.

Equating two consecutive terms we have

$$\frac{n(n-1)(n-2)(n-3)\cdots(n-k)}{(k+1)!}P^{n-k-1}Q^{k+1} = \frac{n(n-1)(n-2)\cdots(n-k-1)}{(k+2)!}P^{n-k-2}Q^{k+2}.$$

Cancelling and multiplying across,

$$P(k+2) = (n-k-1)Q.$$

If now $P = \delta P'$ and $Q = \delta Q'$ where P' and Q' are relatively prime,

$$P'(k+2) = (n-k-1)Q',$$

whence

$$nQ' = 2P' + Q' + k(P'+Q')$$

or

$$(n+1)Q' \equiv 2(P'+Q') \bmod (P'+Q'),$$

that is

$$(n+1)Q' \equiv 0 \bmod (P'+Q'),$$

and since Q' is prime to $P'+Q'$,

$$n+1 \equiv 0 \bmod (P'+Q').$$

That is

$$n \equiv -1 \bmod (P+Q)/\delta.$$

More generally if $n \equiv -1 \bmod (P+Q)/\delta$ there will be two consecutive terms in the expansion having the ratio β/α, δ being the greatest common divisor of αP and βQ.

AN APPLICATION OF PARTIAL DERIVATIVES TO THE ELLIPSE.

By M. O. TRIPP, Yonkers, N. Y.

The object of this article is to show how we may make use of partial derivatives in graphing and discussing the properties of a given ellipse.

Let us take the equation of an ellipse, in rectangular coördinates, in the general form

$$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \quad (h^2 - ab < 0),$$

from which we obtain

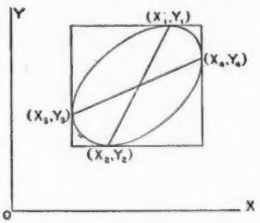
$$\frac{\partial f(x, y)}{\partial x} = 2ax + 2hy + 2g, \quad \text{and} \quad \frac{\partial f(x, y)}{\partial y} = 2hx + 2by + 2f.$$

Making use of the formula

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f(x, y)}{\partial x}}{\dfrac{\partial f(x, y)}{\partial y}}$$

we have, for the slope of the tangent at any point on the ellipse,

$$\frac{dy}{dx} = -\frac{ax + hy + g}{hx + by + f}. \tag{1}$$



At the highest and lowest points of the curve, namely, (x_1, y_1) and (x_2, y_2) respectively, $dy/dx = 0$, since the geometrical tangents at these points are parallel to the x-axis. Hence we observe from (1) that the equation

$$ax + hy + g = 0 \tag{2}$$

is satisfied by the coördinates of the points (x_1, y_1) and (x_2, y_2), since the denominator in the right member of (1) cannot become infinite for these values. Thus it follows that (2) is the equation of the diameter joining (x_1, y_1) and (x_2, y_2); it is the equation of the diameter bisecting the chords parallel to the axis of x.

At the points farthest to the left and right on the curve, namely, (x_3, y_3) and (x_4, y_4) respectively, $dx/dy = 0$, since the geometrical tangents at these points are perpendicular to the x-axis.
Hence

$$hx + by + f = 0, \tag{3}$$

since the numerator of the right member of (1) cannot become infinite for the values considered.

Hence (3) represents the equation of the diameter joining (x_3, y_3) and (x_4, y_4); (3) is the equation of the diameter bisecting chords parallel to the axis of y.

It follows that the coördinates of the center of the ellipse are given by solving simultaneously the equations (2) and (3), or what amounts to the same thing, the equations

$$\frac{\partial f(x, y)}{\partial x} = 0 \tag{4}$$

and

$$\frac{\partial f(x, y)}{\partial y} = 0. \tag{5}$$

It is obvious that all of the methods described above may be applied to any conic

$$ax^2 + 2bxy + cy^2 + 2gx + 2fy + c = 0.$$

If any failure results, it is certain that the conic is a special form for which the corresponding concept does not exist. Thus the usual rule for determining whether (6) is or is not a parabola results from the test whether (4) and (5) have or have not a common solution.

The tests for highest and lowest points apply, with obvious restrictions, to any curve whatever.

BOOK REVIEWS.

UNDER THE DIRECTION OF W. H. BUSSEY.

Algebra, First Course. By EDITH LONG and W. C. BRENKE. The Century Co., New York, 1913. viii+283 pages.

This is the first of a series of texts of correlated mathematics for secondary schools and, in particular, it is the first year's work of a two years' course in algebra and geometry. In its preparation, the authors had in mind two purposes. The first was, in their own words, "to vivify the treatment of algebra by a systematic correlation with geometry." The other was to present the matter in a more simple narrative style than is done in most texts.

To carry out the first purpose, the authors have introduced some of the simpler theorems of geometry and the measurement of geometric quantities as well as some of the fundamental concepts of physics. The proofs of the theorems are in the nature of experimental verification. The introduction of such new matter will undoubtedly stimulate the interest of many students but it is a question if the appeal will be as wide as some have thought. It is evident that a class using this text cannot become as proficient in handling ordinary algebraic operations as is possible with other books. This is more than offset by the gain in the second year's work.

The topics treated are generally well and clearly handled. Where possible, the authors have made use of graphic methods to illustrate the principles. The simple style lends itself well to explanations. In definitions there is danger that it will lead to indistinct or even false conceptions, as when an angle is described as a figure formed by two straight lines starting from the same point.

The authors have requested that teachers using the book omit as little of the text as possible. This request merely points out the fact that the teacher as well as the student will need to be educated in its use.

R. R. SHUMWAY.

An Elementary Treatise on Calculus. By WILLIAM S. FRANKLIN, BARRY MACNUTT and ROLLIN L. CHARLES. Published by the authors, South Bethlehem, Pa., 1913. x+253+41 pages. $2.00.

This book, styled in the subtitle "a textbook for colleges and technical schools," is a noteworthy contribution to the texts carrying out the "Perry movement" in this country. The outstanding feature of the book is expressed in the preface in the purpose of the authors "to lead the student to a clear understanding of principles" and "to develop the subject as simply and as directly as possible." And yet it is exactly at this point that the reviewer's chief criticism is raised; this is in regard to the introduction of infinitesimals as indicated by the following quotation from page 23 of the text:

"In many cases, however, it is desirable to find the limiting relation between Δy and Δx when Δx approaches zero *without deriving an expression for* $\Delta y/\Delta x$. For example let $y = ax^3$. Then

$$\Delta y = 3ax^2 \cdot \Delta x + 3ax(\Delta x)^2 + a(\Delta x)^3. \qquad (1)$$

Both members of this equation approach zero when Δx approaches zero, and it would therefore seem rather difficult to determine the limiting relation between Δy and Δx (when Δx approaches zero) from this equation as it stands. But when Δx is made as small as you please, then $(\Delta x)^2$ is infinitely smaller than Δx, and $(\Delta x)^3$ is infinitely smaller than $(\Delta x)^2$. For example, let Δx be a [millionth of a unit, ... Therefore the terms $3ax(\Delta x)^2$ and $a(\Delta x)^3$ become more and more nearly negligible in comparison with $3ax^2\Delta x$ as Δx grows smaller and smaller in equation (1). The *limiting relation* between Δy and Δx may be found by writing dx and dy for Δx and Δy to indicate that we have proceeded to the limit, and by dropping every term which contains the square or any higher power of Δx (or the square or any higher power of Δy). This gives

$$dy = 3ax^2 \cdot dx. \qquad (2)$$

In this equation dy and dx are as small as you please and they are called *infinitesimals;* . . . ."

In their preface the authors disclaim any intention of taking sides in the old controversy about the method of limits and that of infinitesimals, and they acknowledge there that the treatment here given is fallacious. The question then arises fairly and squarely whether they as textbook makers are justified, on grounds of simplicity or practical purposes or for any such reason, in using an analysis so inexact in the presentation of the fundamental ideas of the calculus. The writer is frank to say that in his judgment they are not justified. Beginners in calculus grasp the technique of differentiating and integrating rather than the analytic work involved in the proofs of the formulas and in the opening discussions, which latter should give *exact* notions of the calculus; and it surely does not sharpen these immature minds to read a text which with only a slight explanation comes at once to such steps as are given in the last two sentences quoted. The development much to be preferred *for all students* is that shown by the following excerpts from the 1912 "Syllabus of Mathematics" compiled by the committee on the teaching of mathematics to students of engineering under the direction of the Society for the Promotion of Engineering Education:[1]

"The ratio $\Delta y/\Delta x$ may be called the *average rate of change of the function during the interval from* $x = x_0$ *to* $x = x_0 + \Delta x$. (Geometrically, $\Delta y/\Delta x$ is the slope of the secant.) Now let Δx

[1] Copies can be obtained from the Secretary, Professor H. H. Norris, Ithaca, N. Y.

approach zero. . . . Then *the ratio $\Delta y/\Delta x$ will in general approach a definite limit, and this limit is called the actual rate of change at the point $x = x_0$* (geometrically, the limit of $\Delta y/\Delta x$ is the slope of the tangent).

"The *rate of change* of a function $y = f(x)$ at any point, or the *slope of the curve* at that point, is called the *derivative* of the function at that point, and is denoted by $f'(x)$, ... The value that Δy *would have* if the curve coincided with its tangent is called the *differential* of y. . . . The use of differentials gives us a *new notation for the derivative*, $f'(x) = dy/dx$."

It should be said that in the last statement quoted it is implied, but was unfortunately not stated clearly, that the right-hand member is the quotient of two differentials.

To return to the text under review, the first chapter—a general survey of differential and integral calculus—is ably, because concretely, developed, presenting successively the notions of rate of change, formal derivative, differential, functions which have the same derivative, work required to stretch a spring, area under a parabola, differential equations, the definitions of indefinite integral and definite integral, with the symbol for the last. Inasmuch as about fifty problems are listed in this introductory chapter, it will be seen that the basic principles of both branches of the calculus need to be fairly mastered at the outset of the course. It would be a fair question to raise whether it is not crowding the student beyond his ability to give him all these ideas in one chapter, when, as we all know, the single abstract notion of differentiation is of such difficulty to many in our classes that it is only slowly grasped—*actually grasped* and handled as a working tool—by the beginner. If on the other hand it is not meant that these new notions should all be mastered at once, why present them all in one chapter? A weakness in Art. 7 may be noted here. Where the *temperature gradient* is defined, the argument is made that the temperature T along a rod must vary *continuously*, because "otherwise the temperature gradient of T at a point would be unthinkable." Only if the students were already acquainted with the idea of gradient, would this argument be convincing.

Chapter II develops in the usual fashion the standard formulas for differentiation. These are referred to in the next chapter, on "Integration," as rules for integration also. In addition, applications of these are made to velocity and acceleration, harmonic motion, curvature, and, as examples of geometric differentiation, to the acceleration of a particle moving in a circular orbit, and to the inward force per unit of length of a barrel hoop, with a commendable emphasis throughout on the proper units in which certain distances, velocities, accelerations, etc., should be expressed. See, for example, problem 1, page 36, and problem 10, page 58.

Still further evidence of the concrete character of the text is given by the very novel introduction in Chapter III of the planimeter to explain the idea of integration, as well as by reference to the cyclometer, electricity meter, and car "speed-time" diagram. Indeed, so completely is the emphasis upon the practical side that the whole technique of integration is disposed of in four pages, by referring to integration by inspection, by the aid of four of the simplest of the usual rules for integration, and, beyond this, by the use of the fairly complete

integral tables given at the end of the book. Of course, practice in these is given through a classified collection of some sixty-nine problems. In order to present an unbroken discussion of the principles, the problems for the second and third chapters are relegated to an appendix.

Chapter IV treats of partial differentiation and integration. Partial derivatives are presented by a clever use of the device of a hill built upon the xy-plane, this device being used consistently throughout the explanation of "gradient in two dimensions" and volumes and surfaces as double integrations.

Chapter V, on "Miscellaneous Applications," combines the topics of probable errors of derived results (applying partial derivatives), maximum and minimum values, problem of the bent beam, average value of a function, center of gravity and of pressure, and moment of inertia. The last includes the proof of the principle of parallel axes, but supplies only two problems to which this may be applied; the abbreviated measure of practice problems accorded to this and other important principles is evidently to be explained by the unusually extensive range of subjects transferrred to this course from the second and even third course in calculus, as traditionally arranged.

In consonance with the general method adopted for the book, Chapter VI gives a "plausible" proof of Maclaurin's theorem and from this derives Taylor's theorem, but evinces a desire to make no distinction between the two. The reviewer cannot but regard this as unwise, the former being universally used as a separate form, even though it be a special case of the latter. This chapter presents also Demoivre's theorem, hyperbolic functions, and Euler's equations which express $\sin x$ and $\cos x$ in exponential form. Except for illustrating the utility of these equations for the electrical engineer, the integration of $\sin mx \cos nx$ might better be carried out by means of the substitution:

$$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B).$$

Chapter VII describes certain ordinary differential equations, presents several well chosen illustrations from physics of the "principle of superposition," represented mathematically in the solutions of linear differential equations with constant coefficients; and, under the figure of such familiar examples as the starting and stopping of a boat and the undamped and damped oscillations of a weighted coiled spring, points to the fundamental problems of the theory of alternating currents.

Chapter VIII treats skilfully and clearly the partial differential equation of wave motion, applying it by the method of infinitesimals to the vibrations of a stretched string, and leading of course to an outline treatment of Fourier's sine and cosine series, the proofs as to the convergence and equivalence of the series being waived. That the gist of Fourier's series should be condensed into eight brief pages and in a manner that would seem to appeal readily to the *practical* thinker, is truly noteworthy; it is a happy example of "the new mathematics."

The final Chapter IX startles the reader still more with as simple and sensible a treatment of vector analysis as the writer remembers ever to have seen. True,

much that is in the chapter must be mastered through the instructor's exposition, but we must make proper recognition when we meet with a treatment of vectors that continually keeps before the student the knowledge that vectors *can be used* and that makes the theorem that the curl of a gradient is zero appear as a matter of common sense (even to a pure mathematician). This is done by means of a well-explained example of a curl. The chapter closes with suggestive, rather than complete, proofs of Stokes's theorem and the formulas for small displacements, with incidental discussion of vector fields and potential.

An appendix provides a careful but broad selection and description of texts to be recommended for the student's further study in mathematics and mathematical physics.

WILLIAM DEW. CAIRNS.

PROBLEMS AND SOLUTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

PROBLEMS FOR SOLUTION.

ALGEBRA.

398. Proposed by R. D. CARMICHAEL, Indiana University.

In the equation $x^3 + \alpha x + \beta = 0$, α is an integer divisible by p^2 and β is an integer divisible by p, p being a prime number. Prove that β is divisible by p^2 if the equation is reducible.

399. Proposed by W. H. BUSSEY, University of Minnesota.

A borrows from B \$1,500 and pays back \$34 a month for 63 months. If the last payment closes the account, what rate of interest has A been paying?

400. Proposed by C. N. SCHMALL, New York City.

Sum the series

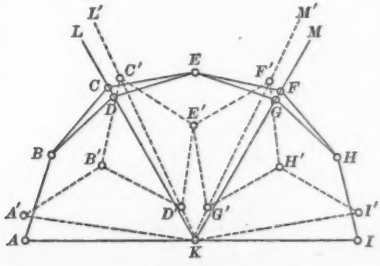
$$1 + 2x + 3x^2 + 4x^3 + \cdots$$

(BROMWICH, *Infinite Series*, p. 129, ex. 1.)

GEOMETRY.

427. Proposed by F. CAJORI, Colorado College.

In S. Gross's linkage for trisection of angles, shown in the figure (KL' and KM' being the



trisectors of $A'KI'$), C is fixed on KL, also F on KM; at starting, C and D coincide, also F and G;

D slides on KL, G slides simultaneously on KM; if E moves along a perpendicular to AI erected at K, find the loci of B and D.[1]

428. Proposed by R. D. CARMICHAEL, Indiana University.

On a given chord of a circle as a base construct a right triangle with vertex outside of the circle such that its hypotenuse shall be bisected by its point of intersection with the circle. Are ruler and compasses sufficient to construct a triangle whose hypotenuse shall thus be divided in any ratio whatever?

429. Proposed by JOHN A. BIGBEE, Little Rock, Ark.

In the trihedral angle V-ABC, the face angle AVB is bisected by the straight line VD. Is it true that the angle DVC is less than, equal to, or greater than, half the sum of the angles AVC and BVC, according as $\angle CVD$ is less than, equal to, or greater than 90°?

430. Proposed by DANIEL KRETH, Wellman, Iowa.

The distance between A and B is always a feet. A travels along a straight path at the rate of v_1 miles per hour, and B starts at the same time in the path behind A and travels in a curve at the rate of v_2 miles per hour. How far will B travel to reach the path in front of A, and how far to reach the path again behind A?

CALCULUS.

350. Proposed by R. P. BAKER, University of Iowa.

Find a general formula for $\frac{d^n y}{dx^n}$ in terms of $\frac{d^k y}{dt^k}$ and $\frac{d^k x}{dt^k}$.

351. Proposed by C. N. SCHMALL, New York City.

In the ellipse $(x^2/a^2) + (y^2/b^2) = 1$, are given the eccentricity e and the angle ϕ which the normal at any point P (on the curve) makes with the major axis. If R is the radius of curvature at P, prove that

$$R = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi)^{3/2}}.$$

352. Proposed by RICHARD P. LOCHNER, Philadelphia, Pa.

At point P there are n foxes. At Q, a rods south of P, there is a dog. The dog and the foxes are freed at the same instant and run at uniform speeds. Some of the foxes run east, some north, some west and some south. The dog runs first toward the foxes that ran east and always points toward them. He captures one of them and then instantly pursues the pack that ran north. In like manner, when he has captured one of them, he pursues those that ran west, then those that ran south, and then begins over again by pursuing the ones running east. If r is the ratio of the dog's speed to that of a fox, what is the total length of the n curves of pursuit?

(Generalization of a problem published in 1859 in the *Mathematical Monthly*.)

MECHANICS.

284. Proposed by C. N. SCHMALL, New York, N. Y.

A cylindrical vessel standing upright on a horizontal plane is kept constantly full of water by an automatic device. Determine at what height in its side a small orifice should be made, so that the water may spout through it to the greatest horizontal distance on the plane. What difference in the result when the cylinder stands on a shelf of known height above the plane?

Note. This problem may have an application in the case of automatic sprinklers.

285. Proposed by RICHARD P. LOCHNER, Philadelphia, Pa.

A ladder 25 ft. long, weighing 120 lb., leans against a vertical wall. Its foot is prevented from slipping on the plane by a peg driven into the ground 7 ft. from the wall. If a man weighing 150 lb. is one-third the way up the ladder, what is the reaction on the peg, the ground, and the wall?

[1] Linkages for the trisection and multisection of angles are described in *Verhandl. des 3. Internationalen Mathematiker-Kongress* in Heidelberg, Leipzig, 1905, pp. 492–496; *Zeitschr. f. Math. u. Physik*, Vol. 49, 1903, pp. 342–347; *Mémoires de la société r. d. sciences de Liége*, 2. série, T. XX, 1898; *Ueber spezielle alg. und trans. Kurven*, von G. Loria (uebers. v. Schütte), Leipzig, 1902, *Sextrix-Kurven*, pp. 316–323.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

203. Proposed by R. D. CARMICHAEL, Indiana University.

Find solutions in integers of the equation $2x^3 + 1 = 3y^2$.

204. Proposed by E. T. BELL, New York City.

Show that a necessary and sufficient condition that $6n + 1$ be a prime number is that no one of the quantities $(3n - r)/(2r + 1)$ for $r = 1, 2, 3, \cdots n - 1$ be an integer; similarly for $6n - 1$, the quantities being $(3n - r)/(2r - 1)$ for $r = 2, 3, 4, \cdots n$.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

389. Proposed by W. W. BEMAN, University of Michigan.

If $e^{e^x} = 1 + a_1x + a_2x^2 + a_3x^3 + \cdots$, prove that

$$na_n = \sum_{k=1}^{k=n} \frac{1}{(k-1)!} a_{n-k} \quad \text{or} \quad n!\, a_n = \sum_{k=1}^{k=n} \frac{(n-1)!}{(k-1)!} a_{n-k}.$$

SOLUTION BY A. M. HARDING, University of Arkansas.

We shall assume that the series

$$e^{e^x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

may be differentiated term by term. We then obtain

$$e^x \cdot e^{e^x} = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} + \cdots$$

or

$$\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots\right)\left(a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + \cdots\right) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} + \cdots.$$

Multiply the two series and equate coefficients of like powers of x. We then have

$$a_1 = a_0,$$

$$2a_2 = a_1 + \frac{a_0}{1!},$$

$$3a_3 = a_2 + \frac{a_1}{1!} + \frac{a_0}{2!},$$

$$4a_4 = a_3 + \frac{a_2}{1!} + \frac{a_1}{2!} + \frac{a_0}{3!},$$

$$\cdots = \cdots$$

$$na_n = a_{n-1} + \frac{a_{n-2}}{1!} + \frac{a_{n-3}}{2!} + \frac{a_{n-4}}{3!} + \cdots \frac{a_0}{(n-1)!}$$

$$= \sum_{k=1}^{k=n} \frac{1}{(k-1)!} a_{n-k}.$$

Solved similarly by A. L. MCCARTY and ELMER SCHUYLER.

390. Proposed by E. B. ESCOTT, University of Michigan.

Sum the series

$$\frac{1}{1}+\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{5}{16}+\frac{8}{32}+\cdots,$$

where each numerator is the sum of the two preceding and the denominators are in geometrical progression.

SOLUTION BY R. D. CARMICHAEL, Indiana University.

The series may be written in the form

$$S = 1 + \sum_{n=1}^{\infty} \frac{u_n}{2^n}, \tag{1}$$

where

$$u_{n+2} - u_{n+1} - u_n = 0, \qquad u_1 = 1, \qquad u_2 = 2. \tag{2}$$

We first solve the recurrence relation (2) without restriction of initial values. Substituting ρ^n for u_n, we find that (2) will be satisfied if

$$\rho^2 - \rho - 1 = 0;$$

that is, if ρ has either of the values

$$\rho_1 = \frac{1+\sqrt{5}}{2}, \qquad \rho_2 = \frac{1-\sqrt{5}}{2}.$$

The general solution of (2), without restriction of initial values, is therefore

$$u_n = \alpha\rho_1{}^n + \beta\rho_2{}^n,$$

where α and β are constants.

We must now determine α and β so that $u_1 = 1$ and $u_2 = 2$; that is, so that

$$\alpha\rho_1 + \beta\rho_2 = 1,$$

$$\alpha\rho_1{}^2 + \beta\rho_2{}^2 = 2.$$

Thus we have readily

$$\alpha = \tfrac{1}{10}(5+\sqrt{5}), \qquad \beta = \tfrac{1}{10}(5-\sqrt{5}).$$

Putting in (1) the value of u_n thus determined, we have

$$S = 1 + \tfrac{1}{10}(5+\sqrt{5})S_1 + \tfrac{1}{10}(5-\sqrt{5})S_2, \tag{3}$$

where

$$S_1 = \sum_{n=1}^{\infty} \frac{(1+\sqrt{5})^n}{2^n \cdot 2^n}, \qquad S_2 = \sum_{n=1}^{\infty} \frac{(1-\sqrt{5})^n}{2^n \cdot 2^n}.$$

Summing these geometric series, we have

$$S_1 = 2+\sqrt{5}, \qquad S_2 = 2-\sqrt{5}.$$

Substituting in (3), we obtain

$$S = 4,$$

the result sought.

Also solved by H. A. LEVY, A. M. HARDING, J. A. COLSON, LEVI S. SHIVELY, H. C. FEEMSTER, ELMER SCHUYLER, and the PROPOSER.

391. Proposed by C. N. SCHMALL, New York, N. Y.

Show that the roots of the quadratic

$$ax^2 + 2bx + c = 0$$

are imaginary if a, b, c are in harmonic progression and have the same sign.

SOLUTION BY S. W. REAVES, University of Oklahoma.

Since a, b, c are in harmonic progression,

$$b = \frac{2ac}{a+c}.$$

The discriminant of the given quadratic then becomes

$$b^2 - ac = \left(\frac{2ac}{a+c}\right)^2 - ac = -ac\left(\frac{a-c}{a+c}\right)^2,$$

which is negative since a and c have the same sign. The discriminant being negative, the roots are imaginary.

Also solved by WALTER C. EELLS, ELMER SCHUYLER, F. M. MORGAN, H. C. FEEMSTER, J. A. COLSON, and A. M. HARDING.

Additional solutions of 387 were received from R. M. MATHEWS, HORACE OLSON, and G. Y. SOSNOW, after the December issue had gone to press.

Note. We have solutions of all problems proposed during 1913 in this section up to the September number, except 385, which was published in February.

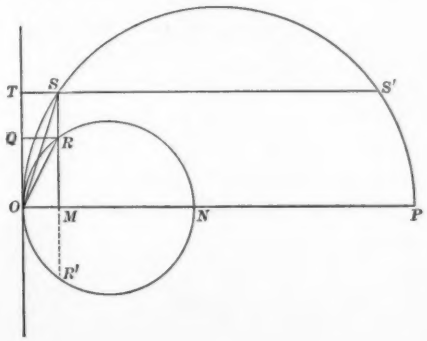
GEOMETRY.

419. Proposed by S. LEFSCHETZ, University of Nebraska.

Given a circle and a tangent to it. To find a point on its circumference such that the sum of its distances to the tangent and its point of contact shall be equal to a given e··h.

SOLUTION BY S. W. REAVES, University of Oklahoma.

Let ORN be the given circle, OT the given tangent, and O the point of contact. Denote the given length by l.



Construction. Draw the diameter ON and extend it beyond N to P making $NP = 2l$. On OP as diameter construct the semi-circle $OSS'P$. On the given tangent lay off $OT = l$. Through T draw the line TSS' parallel to OP to meet the semi-circumference first in S. Drop a perpendicular from S to OP meeting OP in M and meeting the given circle in R and R'. Then R (or R') is the required point.

Proof. Let $QR = OM = x$, $OR = y$, $OS = z$, $ON = 2r$. Then by construction $OP = 2(r + l)$ and $MS = l$.

By elementary geometry $OM : OR = OR : ON$.

Hence

$$y^2 = 2rx. \tag{1}$$

Similarly $OM : OS = OS : OP.$

Hence, $$z^2 = 2x(r + l), \tag{2}$$

or, from (1), $$z^2 = y^2 + 2lx. \tag{3}$$

But $z^2 = x^2 + l^2$, and therefore (3) becomes

$$y = \pm (l - x). \tag{4}$$

Since $y > 0$ and $l > x$, the lower sign is inadmissible. Hence $x + y = l$, or

$$QR + OR = l.$$

A solution exists when and only when $l \leqq 4r$, and there is not more than one solution on the same side of the diameter through the point of contact.

Solutions were also received from WALTER C. EELLS, H. C. FEEMSTER, C. N. SCHMALL, ELMER SCHUYLER, and A. H. HOLMES.

420. Proposed by C. N. SCHMALL, New York City.

Four spheres are described so that each touches a face of a given triangular pyramid and the other three faces produced. If the radii of the escribed spheres be r_1, r_2, r_3, r_4, and r be the radius of the inscribed sphere, show that

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{2}{r}.$$

SOLUTION BY HORACE OLSON, Chicago, Illinois.

Let $ABCD$ be a given tetrahedron.

Let the faces ABC, ABD, ACD, and BCD be represented by small letters corresponding to the opposite vertices of the tetrahedron, *i. e.*, by d, c, b, and a respectively.

Let V represent the volume of the tetrahedron $ABCD$.

Let O be the center, and r the radius, of the inscribed sphere.

Let O_1, O_2, O_3, and O_4 be the centers, and r_1, r_2, r_3, and r_4 the radii, of the escribed spheres opposite to A, B, C, and D, respectively.

Connect points O and O_3 with the 4 vertices of the tetrahedron.

Then,

$$V = AOBC + OBCD + OACD + OABD = \frac{ar}{3} + \frac{br}{3} + \frac{cr}{3} + \frac{dr}{3} = \frac{r}{3}(a + b + c + d),$$

and

$$r = \frac{3V}{a+b+c+d}.$$

Also

$$V = AO_3BC + O_3BCD + O_3ACD - O_3ABD = \frac{r_3}{3}(a+b-c+d)$$

and

$$r_3 = \frac{3V}{a+b-c+d}.$$

Similarly,

$$r_1 = \frac{3V}{(-a+b+c+d)}, \quad r_2 = \frac{3V}{(a-b+c+d)}, \quad \text{and} \quad r_4 = \frac{3V}{(a+b+c-d)}.$$

Therefore

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4} = \frac{(-a+b+c+d)+(a-b+c+d)+(a+b-c+d)+(a+b+c-d)}{3V}$$

$$= \frac{2a+2b+2c+2d}{3V} = \frac{2}{r}.$$

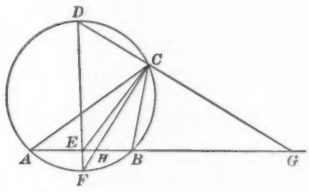
Also solved by A. M. HARDING and ELMER SCHUYLER.

422. Proposed by J. SCHEFFER, Hagerstown, Md.

Construct strictly by use of elementary plane geometry, a triangle having given the product of two sides, the median to the third side, and the difference of the angles to the third side.

SOLUTION BY F. M. MORGAN, Dartmouth College.

Suppose ABC is the required triangle circumscribed by a circle. Draw the internal and external bisectors of $\angle C$ meeting the circle in points F and D. Draw FD.



Since $\angle DCF$ is a right angle, FD is a diameter and bisects AB. Produce DC to meet AB in G. Since $\angle FEG = \angle FCG = 90°$, a circle will pass through points F, E, C, G. Therefore, $\angle EFH = \angle EGC = \frac{1}{2}$ of the difference of the base angles of triangle ABC. This angle is given and will be called α.

Now $\angle EHC = 90° + \alpha$ and is therefore known. Moreover, $CH \cdot CF = CA \cdot CB$ which is a given product.

Construction. Take line CE equal to the given median. Upon it describe the segments of two circles, one to contain an angle α, and the other an angle $90°+\alpha$. Draw line through C cutting these circles so that $CH \cdot CF = CA \cdot CB$. To do

this produce CE to I so that $CE \cdot CI = CA \cdot CB$ and construct an angle at I equal to $90° + \alpha$. Then F is the point where the side of this angle meets the segment containing $\angle\alpha$; for E, H, F, I are on a circle and $CE \cdot CI = CH \cdot CF$.

Draw FE and let it meet the perpendicular to FC at C in point D. Pass a circle through D, C, F and where this circle meets EH extended will be points A and B. The triangle ABC is then the required triangle.

Also solved by A. H. HOLMES, DAVID F. KELLEY, ELMER SCHUYLER, and A. M. HARDING.

A solution of 416 was received from S. W. REAVES too late for notice in the November, 1913, issue.

No solution has been received for 421.

CALCULUS.

A solution of 334 has been received to which the author neglected to sign his name. Two solutions of 327, which was reprinted in the February, 1913, issue, p. 68, have been received. Also solutions are in hand for 336, 339, 341, and 343. All of these will appear in the next issue. Meanwhile we hope to hear from 337, 338, 340, 342, and 344, which include all published up to November, 1913.

MECHANICS.

No solutions have been received for the following problems in mechanics: 246 published in October, 1911; 266 in December, 1911; 268 in January and 269 in April, 1912, and none of those published in 1913, namely 271–283.

Solutions of these problems are desired.

Please remember that problems 276–279 are incorrectly numbered 271–274. See page 258 of the October, 1913, issue.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

190. Proposed by H. C. FEEMSTER.

Show that, if n is a prime number and r an integer less than n, then

$$(r-1)!\,(n-r)! + (-1)^{r-1} = M \cdot n,$$

where M is an integer.

SOLUTION BY THOMAS E. MASON, Indiana University.

From the Wilson theorem we have

$$(n-1)! + 1 \equiv 0 \mod n. \tag{1}$$

We have also,

$$\begin{aligned}(n-1)(n-2)\cdots(n-r+1) &\equiv (-1)(-2)\cdots(-r+1) \mod n.\\ &\equiv (-1)^{r-1}(r-1)! \mod n. \end{aligned} \tag{2}$$

Making substitution from (2) in (1) we get

$$(-1)^{r-1}(r-1)!(n-r)! + 1 \equiv 0 \mod n,$$

or

$$(r-1)!(n-r)! + (-1)^{r-1} \equiv 0 \mod n,$$

which proves the theorem.

Also solved by B. F. YANNEY, L. C. MATHEWSON, and the PROPOSER.

Note. No solutions have been received for problems proposed in 1913 under this heading, except for 188 and 190. Solutions for the remaining ones are desired. Please remember that 191 to 196 are incorrectly numbered 187–192. See page 258 of the October, 1913, issue.

MISCELLANEOUS QUESTIONS.

EDITED BY R. D. CARMICHAEL.

The new department of "Miscellaneous Questions" has met with a pleasing reception among our readers; and we desire to express our appreciation of the interest which has already been manifested through the contributions of questions and replies. We hope to have the coöperation of a considerable number of persons in realizing the double purpose announced in the issue of November, 1913. One of our correspondents, who is a physicist, has written concerning one of these purposes; and we desire to call attention to his remarks in order that we may invite our readers to assist us in realizing the advantages which are mentioned:

"I believe that your department of miscellaneous questions could render valuable service in bringing together the mathematicians and the physicists. Generally the physicist does not know his mathematics sufficiently well to enable him to handle original problems, while the mathematician cannot keep in touch with the developments in physics; but it ought to be possible to work out a plan so that the combined skill of the mathematician and the physicist could be utilized in the solution of the many physical problems. It strikes me that your department of miscellaneous questions could be developed so as to cover this field, and I believe that it would be very useful."

In this issue we have the first replies to our questions. This is the time to say that a question is not disposed of because we have published some answers to it. We wish to have it understood clearly that, although it is desirable to have early replies to our questions, it is yet never too late to send in other replies. In fact, we believe that it will often happen that one set of answers to a given question will provoke others; and that the same question may often be discussed through several issues. Question 2 is one which is probably of this kind.

Again, we shall often print partial answers to a given question in order to call forth a further or more complete answer. This we are doing in the present issue in the case of question 1.

QUESTION.

5. In what ways may mathematics contribute most to the culture of the individual? What is being done and what may be done to advantage in the matter of developing courses in culture mathematics?

REPLIES.

1. In connection with the investigation of a problem in physics Mr. Louis Cohen, Washington, D. C., requires to have expressed in the form of a Fourier's series

$$y = \Sigma A_n \cos nt + \Sigma \beta_n \sin nt \tag{1}$$

the general solution of the differential equation

$$\frac{dy}{dt} + (\alpha + \beta \cos t)y = \rho \cos t, \tag{2}$$

where α, β and ρ are constants. Is the solution of this problem contained directly in matter which has already been published? If not, how may one proceed to the determination of the constants A_n and B_n so that (1) shall be the general solution of (2)?

I. Remarks by C. N. Haskins, Lebanon, N. H.

With reference to the equation in the problem it may be noted that since the general solution is

$$y = \rho(ce^{-(\alpha t+\beta \sin t)} + e^{-(\alpha t+\beta \sin t)} \int e^{\alpha t+\beta \sin t} \cos t dt,$$

a solution in the form

$$y = \rho(ce^{-\alpha t}F_1(t) + F_1(t)F_2(t)),$$

where $F_1(t)$ and $F_2(t)$ are Fourier's series, can be found by noting that the coefficients of the Fourier expansion of $e^{\beta \sin t}$ are simple functions of the quantities $I_n(\beta)$, where

$$I_n(\beta) = i^{-n}J_n(i\beta),$$

the Bessel function of order n with pure imaginary argument. Having found the coefficients in $F_1(t)$ and $F_2(t)$ those in their product may be computed, so that the solution takes the form

$$y = \rho(ce^{-\alpha t}F_1(t) + F(t)),$$

where $F(t)$ and $F_1(t)$ are Fourier's series. This is not the form of solution demanded; but it would appear to be a preferable form for some purposes at least, since the strictly periodic terms are separated from those which are affected with the damping factor $e^{-\alpha t}$ and the arbitrary constant c.

II. Note by the Editor.—The above remarks suggest a means of finding a solution in a form different from that required in the problem. It is given because it seems to be of interest in itself and because further it may be of value in securing the desired solution. We shall be glad to have further communications relative to this problem. Concerning its importance note the following remark by the proposer: "The problem itself is of great importance; a satisfactory solution of this differential equation would lead to a very interesting development of a group of problems in electrical theory."

2. It is clear, on the most casual observation, that the average college curriculum in mathematics is unbalanced in two respects: (1) algebra predominates over geometry, (2) analytic geometry predominates over our synthetic geometry. This reacts in two undesirable ways: (1) to deprive the college student of a rich and interesting field of study, (2) to give a very one-sided training for prospective teachers of high school mathematics. What can be done to remedy this situation?

I. Remarks by D. N. Lehmer, University of California.

In reading the paper on "Synthetic Projective Geometry" in the November issue of the Monthly, I was a little surprised to see that after all our teaching of tha subject in the University of California for at least ten years, with some little

research and very great enjoyment on the part of the students, the work should not have attracted enough notice to be mentioned. We are giving the course to sophomores and even to some freshmen, with and without the calculus, and we think that it has been the salvation of many students. There comes a time in the life of every mathematical student when the continual grind of analysis becomes unbearable. I have felt it myself, and I do not doubt that others have felt it also. Then is the time to give him something really beautiful for a relief. The constant insistence on rigorous proofs and existence theorems does not encourage one in the delusion that the subject of mathematics is one long vista of eternal joy.

I cannot find any particular reason why the calculus should be considered as a "prerequisite" for the subject. If it is supposed that no one can appreciate it without several years of analysis to prepare the way, it only indicates that it has never been tried. I have had freshmen do fine work in the subject and be so filled with the beauty of it that they immediately changed all their plans and went in for mathematics. I have a course in "culture mathematics" and a taste of this study often proves to be the salvation of the course. I think I do not exaggerate when I say that it is the most popular and most thoroughly enjoyed of all the courses given here in mathematics. It is required of all candidates for teachers' certificates in mathematics.

It is gratifying to know that others are waking up to the possibilities of this subject. If, as Mr. Bussey says in his article, the subject is out of the highways of workers in research (and I am not ready to grant that) there is no question of its real value as a relief from the dreary grind of analysis.

II. Remarks by C. E. Stromquist, University of Wyoming.

I have just read the article in the November number of the Monthly on "Synthetic Projective Geometry as an Undergraduate Study." I was very much interested, as I have a class in projective geometry this semester and many of the questions taken up regarding a first course in projective geometry have presented themselves to me. I am not certain whether I agree that a student who is preparing to do graduate work in mathematics "can afford to wait and take projective geometry when he is a graduate." My suggestion would be that he take an introductory course while an undergraduate student in order to be prepared for a more advanced graduate course. But our problem out here in Wyoming is not so much to arrange courses for prospective graduate students as to solve the problem of how to best fit those (undergraduate) students in our mathematical courses who are preparing to teach mathematics in our high schoools. I heartily agree with what Professor Bussey has to say regarding these students and think that his paper is a very timely one.

For a text-book I am using Hatton's "Principles of Projective Geometry" (published by the Cambridge University Press, 1913). In my judgment it is a very satisfactory text for a first course. It covers more ground than one could expect to cover in a half-year course, with three hours a week; but this is not such a serious objection. It contains several sets of problems. There are about 250 of these problems, not including several easier problems placed in the body of the ext.

III. Remarks by T. G. Rodgers, Normal School of New Mexico.

On graduating from a state university several years ago I began teaching high school mathematics. I found that aside from the larger vision gained in undergraduate study in the university I was no better prepared to teach geometry than on completing the high school course. Observation soon convinced me that others too were teaching merely the geometry which they had learned in the high school. An introduction to projective geometry in subsequent graduate work was of benefit but left me still feeling inadequately prepared to teach elementary geometry as a living subject.

My continued search for better methods was finally rewarded through my forming an acquaintance with the works of Rouche and Comberousse, Henrici and Treutlein, and Hadamard. The study of these works has enriched my teaching of geometry to such an extent that I look back with regret to the years in which I tried to teach the subject with only such preparation as the colleges give and with the feeling that college curricula are badly balanced in respect to their preparation of teachers for high school mathematics.

It appears to me that at least one year's work should be given to the study of synthetic geometry, the first half being a lecture course based on works like those mentioned above (especially Hadamard) and the second half being the kind of course in projective geometry outlined by Professor Bussey in the November issue of the Monthly.

NOTES AND NEWS.

Under the direction of Florian Cajori.

Please note that the date on your address label for this issue is changed if your renewal subscription was received before January 10, 1914.

The October number of the *Monist* contains an article by Professor Bertrand Russell on "The Philosophical Importance of Mathematical Logic."

The November number of *School Science and Mathematics* contains an article entitled "A Thread of Mathematical History and Some Lessons," by Professor R. D. Carmichael, of the University of Indiana.

The November number of the *School Review* contains an interesting paper entitled "The Cumulative Examination in Mathematics," by Mr. H. E. Webb, of Newark, N. J. He suggests a series of cumulative examinations extending over the three or four years of high school mathematics.

Dr. Heinrich Burkhardt's "Theory of Functions of a Complex Variable" has been translated into English by Professor S. E. Rasor of the Ohio State University, and published by D. C. Heath & Company.

Spon & Chamberlain have published a book on "The Gyroscope: its Theory and Practise" by Dr. F. J. B. Cordeiro, of the U. S. Navy. While abroad the author made a special study of the gyroscopic compass.

ALBERT W. WHITNEY, formerly associate professor of insurance and mathematics in the University of California, has become assistant actuary in the insurance department of the state of New York.

FRIEDRICH DINGELDEY'S *Sammlung von Aufgaben zur Anwendung der Differential- und Integralrechnung* (B. G. Teubner) contains a collection of problems on the application of the calculus to geometry and also to physics, chemistry and technology.

The Southwestern Section of the American Mathematical Society met at the University of Missouri, Columbia, Mo., on November 29, 1913. There were about twenty members present and twelve papers were presented at the two sessions. Those who read papers were Professors E. R. Hedrick, Louis Ingold, W. C. Brenke, W. D. Westfall, O. D. Kellogg, Drs. E. L. Dodd, S. Lefschetz, A. J. Kempner, H. Blumberg, Mr. A. R. Schweitzer and Miss E. A. Weeks.

The annual meeting of the American Mathematical Society was held in New York on December 30 and 31, 1913.

The American Association for the Advancement of Science held its annual convocation at Atlanta, Ga., during the week beginning December 29, 1913. The retiring president of Section A, Mathematics and Astronomy, was Professor E. B. Van Vleck, of the University of Wisconsin, who gave his address at this meeting on the "Influence of Fourier's Series on Mathematics." The new president is Professor Frank Schlesinger, of Allegheny Observatory, and the secretary is Professor F. R. Moulton, of the University of Chicago.

The Northeastern Ohio Teachers' Association met at Cleveland, October 24 and 25. At the departmental meeting on mathematics papers were read as follows: "How to make the transition from arithmetic to algebra," "What should be included and what excluded in first-year algebra," "Some specific suggestions for teaching the pupil how to attack originals in geometry."

Professor CARLO BOURLET of L'École Nationale, Paris, died August 12, as the result of injuries received in an accident. Aside from notable work as a lecturer in higher mathematics, he has contributed in an important way to the teaching of elementary subjects, chiefly through texts in both algebra and geometry.

The American Mathematical Society has accepted the invitation of Brown University, extended through the committee on the celebration of her 150th anniversary, to hold the summer meeting at Providence in September, 1914.

The first part of Tome II, Volume 6, of the *Encyclopédie des Sciences Mathématiques*, was published on September 15, 1913. This fascicule of 128 pages is devoted to calculus of variations, and was edited by M. Lecat. It it interesting to observe that a considerable part of the space is devoted to a consideration of the results obtained by American mathematicians. Foremost among these are the contributions by Professor OSCAR BOLZA, who returned to Germany a few years ago after having done very much to advance American mathematics, both

by his own work during his twenty years of residence here, and also by the inspiration which he gave to a large number of young investigators.

The *Mathematical Gazette* for July and October, 1913, contains an article by SYDNEY LUPTON on "The radix method" of computing logarithms, which possesses certain advantages when logarithms and antilogarithms are required to many places and, in default of tables, must be calculated. The method requires numbers of the form $(1 + r/10^n)$, with their logarithms to one or two places beyond those actually to be used. The method goes back to Briggs, 1624, and was simplified by Robert Flower, 1771, and further modified in 1786, by George Atwood of "machine" fame. The article contains many historical notes on the computation of logarithms. Several nineteenth century writers rediscovered the old method of Briggs and Flower.

During the Annual High School Conference held at the University of Illinois, November 20, 21, and 22, 1913, the following papers were presented before the mathematics section: "The use of the Courtis tests," by Professor L. D. COFFMAN, University of Illinois; "Statistical method in educational studies," by Professor A. C. LUNN, University of Chicago; "Bibliography on methods of grading," by Mr. H. C. ZEIS, University of Illinois. There was also held a "Round table discussion on methods of grading." According to the program, "All superintendents, principals and high school teachers, as well as teachers in academies, normal schools, colleges and universities are invited to be present and participate in the discussion."

The Central Association of Science and Mathematics Teachers held its thirteenth annual meeting at Des Moines, Iowa, on November 28 and 29, 1913. Professor Florian Cajori was one of the leading speakers at the opening general session and also at one of the sessions of the mathematics section. At the former he spoke on "Science and mathematics in vocational schools; a retrospect," and at the latter on "Means of measuring mathematical abilities." Other numbers on the two programs of the mathematics section were: "Report of a committee on vocational mathematics," by R. L. Short of Cleveland, Ohio; "Report of a committee on examinations and results" by C. E. Comstock of Peoria, Ill., and a paper on "The traditional examination in mathematics" by Jane V. Pollock of Kenilworth, Ill. The report on vocational mathematics was discussed by Professor G. A. Smith, Iowa City, W. Lee Jordan, Des Moines, and W. G. Swartz, Gary, Ind. The association is one of the largest and most active in the secondary field. The mathematics section has put forth at various times some important reports which have had wide influence; in particular, its report on algebra some years ago had a circulation of over ten thousand copies and has without doubt had a marked influence on most of the texts on elementary algebra published since that time. This association was organized in Chicago in 1901 and most of its meetings have been held in Chicago.

The Chicago Section of the American Mathematical Society held its thirty-second regular meeting at the University of Chicago on December 26 and 27,

1913. There were thirty-five papers on the program and 56 members were in attendance. The Chicago section was organized in 1897 and has steadily grown in importance both as to the number of members of the Society attending its meetings, and as to the number of scientific papers presented. The Society itself was organized in New York in 1894, and the regular meetings of the Society are usually held in New York (see a paper on "Western Meetings of Mathematicians" in the April, 1913, issue of the MONTHLY). The Council, however, has recently voted to designate the meetings of the Chicago section as "meetings of the Society at Chicago," and the meeting just held was the first under this designation. In honor of the occasion the University of Chicago tendered a complimentary dinner to the members present.

The *Mathematical Gazette* for October, 1913, contains an exceedingly interesting account, by Dr. C. G. KNOTT, of the Edinburgh mathematical colloquium, held last August, at which three courses of lectures were given, respectively, by an Englishman, E. P. WHITTAKER; an Irishman, A. W. CONWAY; and a Scotchman, D. M. Y. SOMMERVILLE. Conway discoursed on the theory of relativity; Whittaker explained in his mathematical laboratory practical harmonic analysis and periodogram analysis; Sommerville expounded non-euclidean geometry. We quote the following:

"Professor Whittaker chose for his working data the light periods of two variable stars, the one to illustrate the periodigram method of discovering unknown periods, the other to illustrate the analysis into harmonic components of a given periodic variation. The theory of the Fourier analysis was incidentally given; and the last lecture finished with an account of Mäder's Harmonic Analyser."

"This hour of practical work, combined with demonstrations, involving only the familiar circular functions, gave the necessary balance to the weird imaginings of the other two courses. Without it to bring us back to the obvious world of apparent realities we should have been floundering hopelessly in the Absolute or in Minkowski's *Welt*. After we had been taught that velocities did not compound according to the parallelogram law, it was a positive delight to find that the Fourier series remained ordinarily additive; and with this in possession we had no great difficulty in apprehending the possibility of a space devoid of parallel lines. . . ."

"We learned many things. We were told that even if the theory of relativity were not true it had taught us truths. The tendency of modern physical theory was in the direction of still further atomising the atom; yet it was necessary in geometry to have an assumption of continuity, so that all possible numbers might be brought into correspondence with an infinitude of points on a finite line. The dictum of the logician that we cannot define by means of a negation seemed to have no terror to the modern geometer with his glib talk on non-Euclidean, non-Pascalian, non-Desarguesian, and even non-Archimedean."

In the spirit of the editorial statement on the first page of this sisue, the action recently taken by the mathematics teachers of California, as indicated in the following communication, is most significant. The editors of the MONTHLY appreciate the compliment and recognize the responsibility which this action implies. We congratulate the teachers of California upon being pioneers in taking such a bold, forward step.

The communication, under date of January 2, 1914, is from Professor Henry W. Stager, of Fresno Junior College, chairman of the mathematics section of the California High School Teachers' Association, and is addressed to the teachers of mathematics in the secondary schools of California. It reads as follows:

"At the summer meeting of the Mathematics Section of the California High School Teachers' Association, a Committee, composed of Professor D. N. Lehmer, Miss S. Gilmore, and Professor G. A. Miller, was named to consider ways and means of broadening the scholarship of secondary school teachers. The Committee recommended the reading of the book, *Mathematical Monographs*, by J. W. A. Young in collaboration with other mathematicians, and further stated: 'In connection with this book, or even instead of it, the reading of some elementary but strictly first class journal is also recommended. Where it is possible for a number of teachers to meet frequently for the discussion of problems and short articles this plan has much to commend it. For this purpose the Committee would recommend the AMERICAN MATHEMATICAL MONTHLY.'

"The report has been adopted unanimously. The movement for a higher standard of efficiency is one of the strongest attempts to increase the quality of the teaching of mathematics ever made in California. It began with the teachers themselves and only needs your coöperation to make it a success. I commend the report of the Committee to your most earnest consideration.

"At this time I wish especially to call your attention to the reading of a first class journal. I have gone over with care the first volume of THE AMERICAN MATHEMATICAL MONTHLY under the new organization and find it well fitted for this purpose. I am personally acquainted with many members of the Publication Committee and with all the members of the Editorial Committee. The men are in the forefront of the present movement toward a higher standard for the teaching of mathematics in the United States. The MONTHLY represents the best in its field. I commend it to you, feeling sure that it will prove of very great benefit to you, will help to increase your efficiency, and will be a continual source of inspiration to you.

Yours, for the better teaching of mathematics,

HENRY W. STAGER.

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THE ALGEBRA OF ABU KAMIL.

By L. C. KARPINSKI, University of Michigan.

Popularity of Algebra Among the Arabs. The great interest in the study of algebra among the Arabs of the Middle Ages is attested by the numerous works written upon this subject by Mohammedan mathematicians. The period of this popularity extends from the ninth to the fifteenth century. The students of algebra included poets, philosophers, and kings. The popular Omar Khayyam was too excellent a mathematician and too fine a poet to woo the muse of algebra in verse. But his treatise in prose on algebra, edited with French translation by Franz Woepcke, is as good a title to fame as his verse. In the library of the Escurial is preserved a *poem* written six hundred years ago by a native of Granada, Mohammed al-Qasim, and treating of algebra. Needless to say the content does not compare with the prose of Omar Khayyam. The King of Saragossa Jusuf al-Mutamin (reigned 1081–1085) was a devoted student of the mathematical sciences. The title of one of his works would seem to indicate its algebraical nature. Even in the study of law a knowledge of algebra seems to have been necessary, for various questions of inheritance were treated by this science. On the whole, however, in that happy day the question of practical applications of mathematics did not loom so large upon the horizon of the Arabic scientist. That fortunate individual felt entirely free to pursue this subject as one of the natural and inevitable activities of the human intellect.

The First Great Arabic Writer on Algebra. The first systematic treatise on algebra was the product of an Arabic writer, Mohammed ibn Musa, al-Khowarizmi (c. 825 A. D.). Commentaries were written upon this work by Arabic authors and two different Latin translations were made of it. One of these translations was published by Libri in his *Histoire des sciences mathématiques en Italie*, Vol. I, 253–297, Paris, 1838. This work is out of print. While the authorship of this version is uncertain the composition may safely be placed in the twelfth century, as several writers of that century appear to have studied this work and use the terminology of this translation. Manuscript copies are somewhat common in